IHARA LIFTS AND CONJECTURAL CORRESPONDENCES BETWEEN SYMPLECTIC AUTOMORPHIC FORMS OF GENUS TWO (Profinite monodromy, Galois representations, and Complex functions)

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1. Introduction

This article has two aims. Firstly we give a conjectural correspondence of automorphic forms between different $\mathbb{Q}$ forms of symplectic groups of rank two with respect to parahoric subgroups, and secondly we give a precise conjectural images of Ihara lifts (a lift from pairs of elliptic cusp forms to automorphic forms of a $\mathbb{Q}$ form of symplectic group whose archimedean part is compact.) These conjectures are based on dimensional relations of (global) automorphic forms and a lot of explicit examples. Since the contents have been already explained in papers [5], [6], [9], [7], we think there is not much point to repeat it here. So we will only sketch some outline and skip complicated parts. The author hopes interested readers check the papers themselves.

Here we consider the symplectic group $Sp(2,\mathbb{R}) \subset M_4(\mathbb{R})$ and its compact twist $USp(2)$. It is expected by the Langlands conjecture that there should exist a good correspondence between automorphic forms on $Sp(2,\mathbb{R})$ and those on $USp(2)$ preserving the L functions. In case of $SL(2,\mathbb{R})$ and $SU(2)$, the same sort of correspondence is now called Jacquet-Langlands correspondence, but originally such a description were given first by Eichler for concrete discrete subgroups in terms of Brandt matrices for $SU(2)$. Our aim is to generalize this classical Eichler's correspondence to the case of degree two symplectic groups and we are not aiming a description for the whole automorphic representations. This problem was suggested by Y. Ihara around in 1963 before Langlands announced his quite general conjectures. In Ihara's paper [10], he did two things. One is to give a definition of automorphic forms on compact twist $USp(2)$ very concretely and developed an analogy of the classical theory of Brandt matrices. (Such modular forms on algebraic groups such that the archmedean part is a compact group is now called *algebraic modular forms* by B. Gross. See also Hashimoto [4] for a complete description for symplectic case including Hecke algabras.) The other is to prove that under certain conditions, there exist a lift from pairs of elliptic cusp forms to the algebraic modular forms. This can be regarded as a compact version of Saito-Kurokawa lift or Yoshida lift and it is interesting that Ihara lift was obtained much earlier. But there was no conjectures at all for the images of these lifts. Here we can propose a conjecture of images as a by-product of conjectural global correspondence between symplectic automorphic forms belonging to parahoric subgroups.

2. Quaternion hermitian forms and Ihara lift

2.1. **Definition of automorphic forms.** We fix a prime p. We denote by D the definite quaternion algebra over $\mathbb{Q}$ of discriminant p. We fix a maximal order O of D. For any place v of $\mathbb{Q}$, we put $D_v = D \otimes_{\mathbb{Q}} \mathbb{Q}_v$. When v is a finite place, we put $O_v = O \otimes_{\mathbb{Z}} \mathbb{Z}_v$. We define the group G of similitudes of the positive definite binary quaternion hermitian form over D by

$$G = \{g \in M_2(D); gg^* = n(g)1_2, n(g) \in \mathbb{Q}^\times\}.$$

We have

$$G \times_{\mathbb{Q}} \mathbb{C} \cong GSp(2, \mathbb{C}) = \{g \in M_4(\mathbb{C}); gJ\,{}^tg = n(g)1_4, n(g) \in \mathbb{C}^\times\}.$$

We denote by G_A the adelization of G and by G_v the v-component of G_A for any places v of $\mathbb{Q}$. In D^2, there are two genera of quaternion hermitian maximal lattices in the sense of Shimura. One is the genus containing O^2 and we denote this by $\Lambda_{pr}(p)$. We call the other the non-principal genus and we denote this by $\Lambda_{npr}(p)$. For $v \neq p$, the local representatives of both genera are given by O_v^2 and the stabilizer of O_v^2 in G_v is $GL_2(O_v) \cap G_v$, where $GL_2(O_v) = M_2(O_v)^\times$. At p, there are two different local representatives and their stabilzers are representatives of the maximal compact subgroups of G_p up to conjugation. We denote by $U_{pr}(p)$ and $U_{npr}(p)$ the stabilizers in G_A of fixed representatives $L_{pr} = O^2 \in \Lambda_{pr}$ and $L_{npr} \in \Lambda_{npr}$ respectively. We choose L_{npr} so that the components of $U_{pr}(p) \cap U_{npr}(p) = U_{min}(p)$ at p is a minimal parahoric subgroup of G_p. Let (ρ, V) be a irreducible finite dimensional representation of $G_\infty^{(1)} = \{g \in G_\infty; n(g) = 1\}$. We assume that $\rho(\pm 1_2) = id$. We define a representation of G_A associated with ρ by

$$G_A \to G_\infty \to G_\infty/center \cong G_\infty^{(1)}/\{\pm 1_2\} \to GL(V),$$

and denote this also by ρ. We denote by $G_{A,fin}$ the finite part of G_A (i.e. $G_A \cap \prod_{v<\infty} G_v$). For an open compact subgroup U_{fin} of $G_{A,fin} \cap \prod_{v<\infty} G_v$ we define automorphic forms on G_A of weight ρ with respect to subgroup $U = G_\infty U_{fin} \subset G_A$ by

$$\mathfrak{M}_\rho(U) = \{f : G_A \to V; f(uga) = \rho(u)f(g) \text{ for any } g \in G_A, a \in G, u \in U\}.$$

This space is isomorphic to the following space. We take the doucle coset decomposition $G_A = \bigcup_{i=1}^h Ug_iG$ and put $\Gamma_i = G \cap g_i^{-1}Ug_i$. These are finite groups. We denote by V^{Γ_i} the set of elements $v \in V$ such that $\rho(\gamma)v = v$ for all $\gamma \in \Gamma_i$. Then we have

$$\mathfrak{M}_\rho(U) \cong \oplus_{i=1}^h V^{\Gamma_i}.$$

(cf. [10], [4], [1].) Here for $\rho = det^k Sym(j)$, we can give V more concretely. Let $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ be the Hamilton quaternion algebra over $\mathbb{R}$. We identify $\mathbb{H}$ with $\mathbb{R}^4$ by $x = x_1 + x_2 i + x_3 j + x_3 k \in \mathbb{H} \to (x_1, x_2, x_3, x_4)$. For $\lambda = (\lambda_1, \ldots, \lambda_4) \in \mathbb{H} \cong \mathbb{R}^4$ and $(x, y) = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) \in \mathbb{H}^2 \cong \mathbb{R}^8$, we put

$$\Delta_{x,y} = \sum_{i=1}^{4} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} \right).$$

and

$$\Delta_\lambda = \sum_{i=1}^{4} \frac{\partial^2}{\partial \lambda_i^2}.$$

For any even integers 2ν, we denote by $Harm_{2\nu}$ the space of polynomials $f(x, y)$ in $(x, y) \in \mathbb{H}^2 \cong \mathbb{R}^8$ of 8 variables of degree 2ν such that $\Delta_{x,y} f = 0$. For any integers a, b such that $a \geq b \geq 0$ and $a + b = 2\nu$, we put

$$V_{a,b} = \{f(x, y) \in Harm_{2\nu}; f(\lambda x, \lambda y) = n(\lambda)^b \phi(x, y, \lambda), \Delta_\lambda(\phi) = 0\},$$

where $n(\lambda)$ is the reduced norm of λ, Then we have

$$Harm_{2\nu} = \oplus_{a \geq b \geq 0, a+b=2\nu} V_{a,b}$$

For a non-negative integer l, we denote by τ_l the symmetric tensor representation of $SU(2) \cong \mathbb{H}^1 = \{x \in \mathbb{H}; n(x) = 1\}$. We denote by $\tau_{a,b}$ the irreducible representation of G_∞^1 corresponding to the dominant integral weight (a, b). Here we can define the action of $\mathbb{H}^1 \times G_\infty^1$ on $V_{a,b}$ by $((h, g)f)(x, y) = f((\overline{h}x, \overline{h}y)g)$ for any $h \in \mathbb{H}^1$ and $g \in G_\infty^1$ and we can show that this gives the irreducible representation $\tau_{a-b} \otimes \tau_{a,b}$. So actually $V_{a,b}$ does not give an irreducible representation of G_∞^1 unless $a = b$ since there is a multiplicity which is equal to $\dim \tau_{a-b} = 1 + a - b$. Of course we can choose an irreducible subspace of $V_{a,b}$ of G_∞^1 if we like, but this would not be very natural.

2.2. **Ihara lift.** Roughly speaking, Ihara lift is a lift from pairs of elliptic cusp forms to automorphic forms in $\mathfrak{M}_{k+j-3,k-3}(U)$ for some U. For an automorphic form $f(x, y) \in \mathfrak{M}_{k+j-3,k-3}(U)$, we can construct an elliptic modular form by using quaternon hermitian forms and the harmonic polynomial $f(x, y)$. On the other hand, we can make $SU(2)$ act on $f(x, y)$, so by Eichler this also gives an elliptic modular form of different weight. When these construction do not vanish, we can relate the L function of f with L functions of elliptic modular forms. By Löschel [12], this lift is explained by Howe's dual reductive pair coming from the anti-hermitian form of degree two and G_∞. But we follow here Ihara's original formulation. We assume $U = U_{pr}(p)$ or $U_{npr}(p)$

and $\Lambda = \Lambda_{pr}(p)$ or $\Lambda_{npr}(p)$. For each pr or npr, we write

$$U = \bigcup_{\kappa=1}^{h(U)} U g_i G.$$

We denote by $L = L_1, \ \dots L_{h(U)}$ the representatives of Λ/G. We may write $L_\kappa = L_1 g_\kappa := \cap_{v<\infty}(L_1 g_{\kappa,v} \cap D^2)$. We also define $O_A^\times = D_\infty \prod_{v<\infty} O_v^\times$ and

$$D_A^\times = \bigcup_{i=1}^{h_0} O_A^\times b_i D^\times.$$

We put $O_i = b_i^{-1} O_A^\times b_i \cap D^\times$. For (i,κ) with $1 \le i \le h_0$ and $1 \le \kappa \le h(U)$, we define $L_{i\kappa} = \overline{b_i} L g_i = \overline{b_i} L_\kappa$. We define

$$V_{a,b}^{(i,\kappa)} = \{f \in V_{a,b}; f(\overline{u}(x,y)\gamma) = f(x,y) \text{ for all } (u,\gamma) \in O_i^\times \times \Gamma_\kappa\}.$$

Then the space $\oplus_{i,\kappa,a\ge b\ge 0} V_{a,b}^{(i,\kappa)}$ can be regarded as $\mathfrak{M}_{a-b}(O_A^\times) \times \mathfrak{M}_{a,b}(U)$, where $\mathfrak{M}_{a-b}(O_A^\times)$ is the space of automorphic forms on $D_A^\times$ with respect to $O_A^\times$ of weight τ_{a-b}. By Eichler, this corresponds to elliptic new forms of weight $a-b+2$. For $F = (F_{i\kappa}) \in \oplus_{i,\kappa} V_{a,b}^{(i,\kappa)}$, we define theta series on $\tau \in H_1$ as follows.

$$\vartheta_F^{(i,\kappa)}(\tau) = \sum_{m=0}^{\infty} \sum_{x \in L_{i\kappa}, n_{i\kappa}(x,y)=m} F_{ik}(x,y) e^{2\pi i m\tau}$$

where we put $n_{i\kappa}(x,y) = (n(x)+n(y))/n(L_{i\kappa})$, where $n(L_{i\kappa})$ is the fractional $\mathbb{Z}$ ideal spanned by all $n(x)+n(y)$ for $(x,y) \in L_{i\kappa}$. Then we have $\theta_F^{(i,\kappa)}(\tau) \in A_{a+b+4}(\Gamma_0(p))$ if $U = U_{pr}(p)$ and $\in A_{a+b+4}(SL_2(\mathbb{Z}))$ if $U = U_{npr}(p)$. We put

$$\vartheta_F(\tau) = \sum_{i=1}^{h_0} \sum_{\kappa=1}^{h} \frac{1}{|O_i^\times||\Gamma_\kappa|} \vartheta_F^{i\kappa}(\tau).$$

This is a cusp form unless $a = b = 0$.

Theorem 2.1 ([10],[8]). *Assume that F is a Hecke eigen form in $\mathfrak{M}_{a-b}(O_Z^\times) \times \mathfrak{M}_{a,b}(U)$ and given by $F_1 \times F_2$ ($F_1 \in M_{a-b}(O_A^\times)$, $F_2 \in \mathfrak{M}_{a,b}(U)$. Assume also thet $\vartheta_F \ne 0$. Then ϑ_F is also a Hecke eigenform and we have*

$$L(s,F_2) = L(s-b-1,F_1) L(s,\vartheta_F).$$

If $a \ne b$, then this gives a lift from $S_{a-b+2}^{new}(\Gamma_0(p)) \times S_{a+b+4}(\Gamma_0(p))$ to $\mathfrak{M}_{a,b}(U_{pr}(p))$ and from $S_{a-b+2}^{new}(\Gamma_0(p)) \times S_{a+b+4}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{a,b}(U_{npr}(p))$. This is a compact version of the Yoshida lift in [14]. If $a = b$, then we must add an Eisenstein series to $S_2^{new}(\Gamma_0(p))$ and this case is the compact version of Saito-Kurokawa lift. (Note that in both cases, Ihara's work was done much earlier.) There was no theory on images of this Ihara lift. We propose a conjectural image later.

3. Siegel modular forms and parahoric subgroups

Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two. For any irreducible polynomial representation ρ of $GL(2,\mathbb{C})$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2,\mathbb{R})$ and a function $f(Z)$ on $\mathfrak{H}_2$, we write

$$(f|_\rho[g])(Z) = \rho(CZ+D)^{-1}f(gZ).$$

For a discrete subgroup Γ of $Sp(2,\mathbb{Q})$ with $\mathrm{Vol}(\Gamma\backslash Sp(2,\mathbb{R})) < \infty$, we denote by $S_\rho(\Gamma)$ the space of holomorphic Siegel cusp forms of weight ρ with respect to Γ. Or more precisely, a holomorphic function $f(Z)$ on $\mathfrak{H}_2$ belongs to $S_\rho(\Gamma)$ if

$$f(\gamma Z) = \rho(CZ+D)f(Z) \qquad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$$

and $\Phi(f|_\rho[g]) = 0$ for all $g \in Sp(2,\mathbb{Q})$, where Φ is the Siegel Φ operator. Any irreducible representations ρ of $GL(2,\mathbb{C})$ can be written as $\rho_{k,j} = \det^k Sym(j)$ for some k and j, where $Sym(j)$ is the symmetric tensor representation of degree j, so for $\rho = \rho_{k,j}$, we also write

$$S_\rho(\Gamma) = S_{k,j}(\Gamma).$$

When $j = 0$, we also write $S_k(\Gamma) = S_{k,0}(\Gamma)$. Now we explain discrete subgroups Γ that we will consider later.

The group $Sp(2,\mathbb{Q}_p)$ has seven proper standard parahoric subgoups and corresponding to those, we may define seven discrete subgroups of $Sp(2,\mathbb{R})$. We define them as follows. We put

$$B(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}),$$

$$\Gamma_0(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}).$$

$$\Gamma_0'(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}).$$

$$\Gamma_0''(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}).$$

$$K(p) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & p\mathbb{Z} \\ p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}). \quad Sp(2,\mathbb{Z}) = M_4(\mathbb{Z}) \cap Sp(2,\mathbb{Q})$$

$$Sp^*(2,\mathbb{Z}) = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} & p^{-1}\mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{pmatrix} \cap Sp(2,\mathbb{Q}).$$

If we write

$$\rho = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & p & 0 & 0 \\ p & 0 & 0 & 0 \end{pmatrix}.$$

then we have $Sp^*(2,\mathbb{Z}) = \rho^{-1}Sp(2,\mathbb{Z})\rho$ and $\Gamma_0''(p) = \rho\Gamma_0'(p)\rho^{-1}$. So obviously we have $S_{k,j}(Sp^*(2,\mathbb{Z})) \cong S_{k,j}(Sp(2,\mathbb{Z}))$ and $S_{k,j}(\Gamma_0'(p)) \cong S_{k,j}(\Gamma_0''(p))$. We also note that since $-1_4 \in \Gamma$ for any of the above seven groups, we have $S_{k,j} = 0$ if j is odd since $f(Z) = (f|_{k,j}[-1_4])(Z) = (-1)^j f(Z)$. Here we note that $K(p)$ is the so-called paramodular group of level p, which is an important group for the paradular conjecture on abelian surfaces and also for a theory of new forms.

4. Three dimensional relations

We assume that k, j are non-negative integers such that $k \geq 3$ and j is even. By technical reason, we need some more conditions besides: When $j = 0$, we do not need any more conditions. If $j > 0$ we assume that $k \geq 5$ and $p \neq 2$ or 3. (We believe that the following theorem holds without such technical conditions.)

Theorem 4.1 ([5], [3], [6], [7]). *Under the conditions explained above, we have the following relations of dimensions.*

(1)

$$\begin{aligned} &\dim S_{k,j}(B(p)) - \dim S_{k,j}(\Gamma_0(p)) - \dim S_{k,j}(\Gamma_0'(p)) - \dim S_{k,j}(\Gamma_0''(p)) \\ &\quad + 2S_{k,j}(Sp(2,\mathbb{Z})) + K(p) = \dim \mathfrak{M}_{k+j-3,k-3}(U_{\min}(p)) \\ &\quad - \dim \mathfrak{M}_{k+j-3,k-3}(U_{pr}(p)) - \dim \mathfrak{M}_{k+j-3,k-3}(U_{npr}(p)) + \delta_{k3}\delta_{j0}. \end{aligned}$$

(2)

$$\begin{aligned} &\dim S_{k,j}(K(p)) - 2S_{k,j}(Sp(2,\mathbb{Z})) + \delta_{k3}\delta_{j0} = \dim \mathfrak{M}_{k+j-3,k-3}(U_{npr}(p)) \\ &- (\dim S_{j+2}^{new}(\Gamma_0(p)) + \delta_{j0}) \times \dim S_{2k+j-2}(SL_2(\mathbb{Z})). \end{aligned}$$

(3)

$$\begin{aligned} &\dim S_{k,j}(\Gamma_0'(p)) + \dim S_{k,j}(\Gamma_0''(p)) - \dim S_{k,j}(\Gamma_0(p)) - 2\dim S_{k,j}(K(p)) \\ &= \dim \mathfrak{M}_{k+j-3,k-3}(U_{pr}(p)) - \delta_{j0}\delta_{k3} \\ &- (\dim S_{j+2}^{new}(\Gamma_0^{(1)}(p)) + \delta_{j0}) \times (\dim S_{2k+j-2}^{new}(\Gamma_0^{(1)}(p)) + \dim S_{2k+j-2}(SL_2(\mathbb{Z}))). \end{aligned}$$

Here $\Gamma_0^{(1)}(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}); c \equiv 0 \bmod p \right\}$ and S_*^{new} denotes the space of new forms.

5. Conjectures on Ihara lifts

The meaning of the dimensional relations in the previous section is almost clear for the first and the second one. The first one means that the "new" form belonging to the minimal parahoric should correspond one to one as stated in [2] as a conjecture. For the second relation, we proposed a conjecture in [5], and the space of new forms in $S_{k,j}(K(p))$ are defined there. This is essentially the same as the level p case in [13]. Although the groups $Sp(2,\mathbb{Z})$ and $K(p)$ have no inclusion relation inbetween, still we can define forms in $S_{k,j}(K(p))$ which come from $S_{k,j}(Sp(2,\mathbb{Z}))$. One way to explain this is to say that those forms in $S_{k,j}(K(p))$ obtained by taking the trace of elements of $S_{k,j}(Sp(2,\mathbb{Z})) + S_{k,j}(Sp^*(2,\mathbb{Z}))$ through intersection of discrete subgroups are old forms. Or in the adelic setting, those automorphic representations which have vectors fixed by $Sp(2,\mathbb{Z})$ or $Sp^*(2,\mathbb{Z})$ are old forms. The LHS of the second relation is roughly speaking the dimension of new forms in this sense. But there is a small exception. If k is even, then there exists the Saito-Kurokawa lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $S_k(Sp(2,\mathbb{Z}))$. In this case, the dimension of the corresponding old forms in $S_k(K(p))$ is one, so $-2\dim S_{k,j}(Sp(2,\mathbb{Z}))$ is too much. But from RHS, we have $-\dim S_{2k-2}(SL_2(\mathbb{Z}))$. This calcels with the minus in LHS. On the other hand, if k is odd, then there is no Saito-Kurokawa lift. In this case, $-\dim S_{2k-2}(SL_2(\mathbb{Z}))$ in RHS should be absorpted in the Ihara lift to $\mathfrak{M}_{k-3,k-3}(U_{npr}(p))$ and the contributions to both sides are equally zero. The other minus is also explained in this way. So, also supported by many concrete examples, we proposed the following conjectures in [5], [6].

Conjecture 5.1. *When $U = U_{npr}(p)$, then*
(1) We have an injective Ihara lift from $S_{j+2}^{new}(\Gamma_0(p)) \times S_{2k+j-2}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{k+j-3,k-3}(U_{npr})$.
(2) When k is odd, we have an injective Ihara lift from $S_{2k-2}(SL_2(\mathbb{Z}))$ to $\mathfrak{M}_{k-3,k-3}(U_{npr})$.

Remark. *We are not saying here in this conjecture that a kind of lifts to $S_k(K(p))$ or $\mathfrak{M}_{k-3,k-3}(U_{npr}(p))$ are all obtained in this way. There exists another kind of lift (sometimes called Gritsenko lift and not Ihara lift) to both $S_k(K(p))$ and $\mathfrak{M}_{k-3,k-3}(U_{npr})$ and they correspond with each other. We cannot see this part from the dimensional relation.*

By the way, the images of the Ihara lifts for squarefree level case of U_{npr} is more complicated and has been explained in [9], as well as dimensional relations similar to Theorem 4.1 (2).

The third relation of Theorem 4.1 is much more complicated, but we may consider in the same way. To explain this case, we denote by $S^{new,\pm}(\Gamma_0^{(1)}(p))$ the eigenspace of the Atkin-Lehner involution on $S^{new}(\Gamma_0^{(1)}(p))$ such that the eigenvalue is $+1$ or -1.

We propose following conjectures.

Conjecture 5.2. *(1) When k is odd, then the Ihara lift from $S_{2k-2}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k-3,k-3}(U_{pr}(p))$ is injective. When k is even, there should be no such Ihara lifts.*
(2) There exists an injective Ihara lift from $S_{j+2}^{new,+}(\Gamma_0^{(1)}(p))\times S_{2k+j-2}^{new,-}(\Gamma_0^{1}(p))$ and $S_{j+2}^{new,-}(\Gamma_0^{(1)}(p))\times S_{2k+j-2}^{new,+}(\Gamma_0^{(1)}(p))$ to $\mathfrak{M}_{k+j-3,k-3}(U_{pr}(p))$ and no lift from $S_{j+2}^{new,\epsilon}(\Gamma_0^{(1)}(p)) \times S_{2k+j-2}^{new,\epsilon}(\Gamma_0^{(1)}(p))$ when $\epsilon = 1$ or -1 at the same time for both terms.

These conjectures are supported by a lot of numerical examples and also by a local and global behaviours of various lifts to Siegel modular forms studied by Böcherer-Schulze Pillot and R. Schmidt. For details, please see the paper [7].

6. Ihara's interesting example

Non-lifted part of the relation (3) is complicated. Since any local admissible representation of $GSp(2,\mathbb{Q}_p)$ which has the Iwahori subgroup fixed vector is completely classified by [13], we can explain more in detail of this case, considering together with a lot of results by R. Schmidt, but we omit them here, since they are explained in details in [7] together with numerical examples. Here we only add an interesting Ihara's example in [10]. In his paper, for $p=3$, he gave examples of automorphic forms in $\mathfrak{M}_{\nu,\nu}(U_{pr}(3))$ for $\nu \le 8$, $\nu = 9$ and $\nu = 11$. If $\nu \le 7$, then all the automorphic forms are lifts. He has shown that

$$\dim \mathfrak{M}_{8,8}(U_{pr}(3)) = 6$$

and gave all Hecke eigen basis of $\mathfrak{M}_{8,8}(U_{pr}(3))$. Four of them are lifts. The remaining two are not lifts. Those non-lifts have the same Euler 2 factors (of Spinor L functions), explicitly given by

$$(4)\quad (1-12(-9+\sqrt{1489})2^{-s}+2^{19-2s})(1-12(-9-\sqrt{1489})2^{-s}+2^{19-s}).$$

He suspected that these two forms have the same Euler factors for all $p \neq 3$. On the other hand, by the third dimensional relation in Theorem 4.1, there should exist corresponding Siegel cusp forms belonging to the parahoric subgroups. The corresponding weight in this case is det^{11}. The dimensions $S_{11}(\Gamma)$ is given as follows.

Γ	$\Gamma_0(3)$	$\Gamma_0'(3)$	$\Gamma_0''(3)$	$K(3)$
$dim S_{11}(\Gamma)$	0	2	2	1

(Note here that we mean $\Gamma_0(3) \subset Sp(2,\mathbb{Z}) \subset M_4(\mathbb{Z})$ and not a subgroup of $SL_2(\mathbb{Z})$.) Here the element of $S_{11}(K(3))$ is a lift from the elliptic cusp form of weight 20. Since $S_k(K(p)) \subset S_k(\Gamma_0'(p)) \cong S_k(\Gamma_0''(p))$, one of the form in $S_{11}(\Gamma_0'(3))$ is a lift. The other one is a non-lift and by actual calculation we can show that the Euler two factor is the same as (4). There is a non-lift Siegel cusp form in $S_{11}(\Gamma_0''(3))$ which has the same L function as non-lift og $S_{11}(\Gamma_0'(3))$. Judging from the dimensional relation (3), these two Siegel cusp forms should correspond to two non-lifts in $\mathfrak{M}_{8,8}(U_{pr}(3))$. The non-lifts in $S_{11}(\Gamma_0'(3))$ and $S_{11}(\Gamma_0''(3))$ of course belong to the same automorphic representation for $GSp(4)$. For two forms in $\mathfrak{M}_{8,8}(U_{pr}(3))$, we still do not know if they belong to the same automorphic representation of G_A. If not, this means the counter example for the multiplicity one. By the way, the example of this sort seems not so rare, since we can give more concrete examples similar to this.

We note that there is a case that there is a non-lift Siegel cusp form but no corresponding form in $\mathfrak{M}_{k+j-3,k-3}(U_{pr}(p))$. For example, when $p = 2$ $S_{12}(\Gamma_0(2))$ has two no-lifts and $S_{12}(\Gamma_0'(2))$ and $S_{12}(\Gamma_0''(2))$ have one non-lift respectively (and no non-lift in $S_{12}(K(2))$. The Hecke eigenvalues of these forms are the same at all odd primes, and the dimensional contribution of this part to LHS of Theorem 4.1 (3) is zero. As expected, there is no corresponding form in $\mathfrak{M}_{9,9}(U_{pr}(2))$.

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